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Partially Ordered Metric Space using Fixed Point Theorems

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Abstract

Fixed point theory is an accurate topic of nonlinear analysis. Also, it is well known that the contraction mapping principle. From last four decades, this theorem has undergone various generalizations either by relaxing the condition on contractively or withdrawing the requirement of completeness or sometimes even both. In this paper we stated fixed point theorems and its results analysis for weak C -contractions in a partially ordered metric space.

Keywords: Fixed Point Theory, Coincidence Point, Metric Space.

Introduction

In mathematics, the fixed-point theorem is a result saying that a function F will have at least one fixed point (a point x for which $F(x)=x$, under some conditions on F that can be stated in general terms. Results of this kind are amongst the most generally useful in mathematics. There are in the literature a great number of generalizations of the contraction principle and the references cited therein). In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. In this direction, is addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function. A mathematical object X has the fixed-point property if every suitably well-behaved mapping from X to itself has a fixed point. The term is most commonly used to describe topological spaces on which every continuous mapping has a fixed point. But another use is in order theory, where a partially ordered set P is said to have the fixed point property if every increasing function on P has a fixed point.

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. There exists a last literature on the topic and this is a very active field of research at present. There are great numbers of generalizations of the contraction principle. Many authors extended some fixed point theorems on metric spaces to cone metric spaces. However, it was shown later by various authors that in several cases the fixed point results in cone metric spaces can be obtained by reducing them to their standard metric counterparts. And also have introduced the notion of a complex-valued metric and some fixed point theorems have been stated. Note that in the above generalizations, only a 2 metric space has not been known to be topologically equivalent to an ordinary metric. Then there was no easy relationship between results obtained in 2-metric spaces and metric spaces. In particular, the fixed point theorems on 2-metric spaces and metric spaces may be unrelated easily [1-3] and [6-8].

In this paper we stated some fixed point results for weak C -contractions in a partially ordered 2-metric space.

Partially Ordered Metric Space using Fixed Point Theorems

First we recall few notions and lemmas which will be useful in our theorems analysis as follows:

Definition 1.1: Let X be a non-empty set and let $d: X \times X \times X \rightarrow \mathbb{R}$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ like as $d(x, y, z) \neq 0$.
1. If at least two of three points as x, y, z are the same, then we write as $d(x, y, z) = 0$.
3. The symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$.
4. The rectangle inequality: $d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$.

Then d is called a 2-metric on X and (X, d) is called a 2-metric space which will be sometimes denoted by X if there is no confusion. Every member $x \in X$ is called a point in X .

Definition 1.2: Let (X, d) be a 2-metric space and $a, b \in X, r \geq 0$. The set $B(a, b, r) = \{x \in X : d(a, b, x) < r\}$ is called a 2-ball centered at a and b with radius r . And the topology generated by the collection of all 2 balls as a sub-basis is called a 2-metric topology on X .

Definition 1.3: Let $\{x_n\}$ be a sequence in a 2-metric space (X, d) .

1. $\{x_n\}$ is said to be *convergent* to x in (X, d) , written $\lim_{n \rightarrow \infty} x_n = x$, if for all $a \in X$, $\lim_{n \rightarrow \infty} d(x_n, x, a) = 0$.
1. $\{x_n\}$ is said to be *Cauchy* in X if for all $a \in X$, $\lim_{n, m \rightarrow \infty} d(x_n, x_m, a) = 0$, that is, for each $\varepsilon > 0$, there exists n_0 such that $d(x_n, x_m, a) < \varepsilon$ for all $n, m \geq n_0$.
3. (X, d) is said to be *complete* if every Cauchy sequence is a convergent sequence.

Definition 1.4: A 2-metric space (X, d) is said to be *compact* if every sequence in X has a convergent subsequence.

Lemma 1.7: Every 2-metric space is a T1-space.

Lemma 1.8: (Lemma 4) $\lim_{n \rightarrow \infty} x_n = x$ in a 2-metric space (X, d) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in the 2-metric topological space X .

Lemma 1.9: (Lemma 5) If $T : X \rightarrow Y$ is a continuous map from a 2-metric space X to a 2-metric space Y , then $\lim_{n \rightarrow \infty} x_n = x$ in X implies $\lim_{n \rightarrow \infty} Tx_n = Tx$ in Y .

1. It is straightforward from Definition 1.1 that every 2-metric is non-negative and every 2-metric space contains at least three distinct points.
1. A 2-metric $d(x, y, z)$ is sequentially continuous in one argument. Moreover, if a 2-metric $d(x, y, z)$ is sequentially continuous in two arguments, after then it is sequentially continuous in all three arguments.
3. A convergent sequence in a 2-metric space need not be a Cauchy sequence.
4. In a 2-metric space (X, d) , every convergent sequence is a Cauchy sequence if d is Continuous..
5. There exists a 2-metric space (X, d) such that every convergent sequence is a Cauchy sequence but d is not continuous.

Results Analysis

In this section we analysis the weak C-contraction on a partially ordered 2-metric space.

Definition 1.1 Let (X, \leq, d) be a partially ordered 2-metric space and $T : X \rightarrow X$ be a map. Then T is called a *weak C-contraction* if there exists $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ which is continuous, and $\psi(s, t) = 0$ if and only if $s = t = 0$ such that

$$d(Tx, Ty, a) \leq \frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)] - \psi(d(x, Ty, a), d(y, Tx, a)) \dots\dots\dots(1.1)$$

for all $x, y, a \in X$ and $x \leq y$ or $y \leq x$.

Theorem 1.2 Let (X, \leq, d) be a complete, partially ordered 2-metric space and $T : X \rightarrow X$ be a weak C-contraction such that:

1. T is continuous and non-decreasing
1. There exists $x_0 \in X$ with $x_0 \leq Tx_0$.

Then T has a fixed point.

Proof If $x_0 = Tx_0$, then the proof is finished. Suppose now that $x_0 \leq Tx_0$. Since T is a non-decreasing map, we have $x_0 \leq Tx_0 \leq T^2x_0 \leq \dots \leq T^n x_0 \dots\dots$ Put $x_{n+1} = Tx_n$. Then, for all $n \geq 1$, from (1.1) and noting that x_{n-1} and x_n are comparable, we get

$$\begin{aligned} d(x_{n+1}, x_n, a) &= d(Tx_n, Tx_{n-1}, a) \\ &\leq \frac{1}{2}[d(x_n, Tx_{n-1}, a) + d(x_{n-1}, Tx_n, a)] - \psi(d(x_n, Tx_{n-1}, a), d(x_{n-1}, Tx_n, a)) \\ &= \frac{1}{2}[d(x_n, x_n, a) + d(x_{n-1}, x_{n+1}, a)] - \psi(d(x_n, x_n, a), d(x_{n-1}, x_{n+1}, a)) \\ &= \frac{1}{2}[d(x_{n-1}, x_{n+1}, a) - \psi(0, d(x_{n-1}, x_{n+1}, a))] \\ &\leq \frac{1}{2}[d(x_{n-1}, x_{n+1}, a)] \dots\dots\dots(1.2) \end{aligned}$$

for all $a \in X$. By choosing $a = x_{n-1}$ in (1.2), we obtain $d(x_{n+1}, x_n, x_{n-1}) \leq 0$, that is,

$$d(x_{n+1}, x_n, x_{n-1}) = 0 \dots\dots\dots(1.3)$$

It follows from (1.2) and (1.3) that

$$\begin{aligned} d(x_{n+1}, x_n, a) &\leq \frac{1}{2} d(x_{n-1}, x_{n+1}, a) \\ &\leq \frac{1}{2}[d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, x_{n+1})] \\ &= \frac{1}{2}d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a) \dots\dots\dots(1.4) \end{aligned}$$

It implies that

$$d(x_n, x_{n+1}, a) \leq d(x_{n-1}, x_n, a) \dots\dots\dots(1.5)$$

Thus $\{d(x_n, x_{n+1}, a)\}$ is a decreasing sequence of non-negative real numbers and hence it is convergent. Let

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}, a) = r \dots\dots\dots(1.6)$$

Taking the limit as $n \rightarrow \infty$ in (1.4) and using (1.6), we get

$$r \leq \lim_{n \rightarrow \infty} \frac{1}{2} d(x_{n-1}, x_{n+1}, a) \leq \frac{1}{2} (r + r) = r.$$

That is,

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_{n+1}, a) = 2r \dots\dots\dots(1.7)$$

Taking the limit as $n \rightarrow \infty$ in (1.2) and using (1.6), (1.7), we get $r \leq \frac{1}{2} 2r - \psi(0, 2r) \leq \frac{1}{2} 2r = r$. It implies that $\psi(0, 2r) = 0$, that is, $r = 0$. Then (1.6) becomes

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n, a) = 0 \dots\dots\dots (1.8)$$

From (1.5), we have if $d(x_{n-1}, x_n, a) = 0$, then $d(x_n, x_{n+1}, a) = 0$. Since $d(x_1, x_2, x_3) = 0$, we have $d(x_n, x_{n+1}, x_0) = 0$ for all $n \in \mathbb{N}$. Since $d(x_{n-1}, x_m, x_m) = 0$, we have

$$d(x_n, x_{n+1}, x_m) = 0 \dots\dots\dots (1.9)$$

for all $n \geq m-1$, For $0 \leq n < m-1$, noting that $m-1 \geq n+1$, from(1.9) we have

$$d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0.$$

It implies that

$$\begin{aligned} d(x_n, x_{n+1}, x_m) &\leq d(x_n, x_{n+1}, x_{m-1}) + d(x_{n+1}, x_m, x_{m-1}) + d(x_m, x_n, x_{m-1}) \\ &= d(x_n, x_{n+1}, x_{m-1}) \dots\dots\dots (1.10) \end{aligned}$$

Since $d(x_n, x_{n+1}, x_{n+1}) = 0$, from(1.10) we have

$$d(x_n, x_{n+1}, x_m) = 0 \dots\dots\dots (1.11)$$

for all $0 \leq n < m-1$. From (1.9) and (1.11), we have $d(x_n, x_{n+1}, x_m) = 0$ for all $n, m \in \mathbb{N}$. Now, for all $i, j, k \in \mathbb{N}$ with $i < j$, we have $d(x_{j-1}, x_j, x_i) = d(x_{j-1}, x_j, x_k) = 0$. Therefore,

$$\begin{aligned} d(x_i, x_j, x_k) &\leq d(x_i, x_j, x_{j-1}) + d(x_i, x_k, x_{j-1}) + d(x_k, x_i, x_{j-1}) \\ &\leq d(x_i, x_{j-1}, x_k) \\ &\leq \dots\dots\dots \\ &\leq d(x_i, x_i, x_k) = 0. \end{aligned}$$

This proves that for all $i, j, k \in \mathbb{N}$

$$d(x_i, x_i, x_k) = 0 \dots\dots\dots (1.12)$$

In what follows, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ where $n(k)$ is the smallest integer such that $n(k) > m(k) > k$

And

$$d(x_{n(k)}, x_{m(k)}, a) \geq \varepsilon \dots\dots\dots (1.13)$$

for all $k \in \mathbb{N}$. Therefore,

$$d(x_{n(k)-1}, x_{m(k)}, a) < \varepsilon \dots\dots\dots (1.14)$$

By using (1.12), (1.13) and (1.14), we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}, a) \\ &\leq d(x_{n(k)}, x_{n(k)-1}, a) + d(x_{n(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, x_{n(k)-1}) \\ &= d(x_{n(k)}, x_{n(k)-1}, a) + d(x_{n(k)-1}, x_{m(k)}, a) \\ &< d(x_{n(k)}, x_{n(k)-1}, a) + \varepsilon \dots\dots\dots (1.15) \end{aligned}$$

Taking the limit as $k \rightarrow \infty$ in (1.15) and using (1.8), we have

$$\lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}, a) = \lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{m(k)}, a) = \varepsilon \dots\dots\dots (1.16)$$

Also, from (1.12), we have

$$\begin{aligned} d(x_{m(k)}, x_{n(k)-1}, a) &\leq d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) + d(x_{m(k)}, x_{n(k)-1}, x_{m(k)-1}) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) \\ &\leq d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)}, a) + d(x_{n(k)-1}, x_{n(k)}, a) + d(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)}) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)}, a) + d(x_{n(k)-1}, x_{n(k)}, a) \dots\dots\dots (1.17) \end{aligned}$$

And

$$\begin{aligned} d(x_{m(k)-1}, x_{n(k)}, a) &\leq d(x_{m(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, a) + d(x_{m(k)-1}, x_{n(k)}, x_{m(k)}) \\ &= d(x_{m(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, a) \dots\dots\dots (1.18) \end{aligned}$$

Taking the limit as $k \rightarrow \infty$ in (1.17), (1.18) and using (1.8), (1.16), we obtain

$$\lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)}, a) = \varepsilon \dots\dots\dots (1.19)$$

Since $n(k) > m(k)$ and $x_{n(k)-1}, x_{m(k)-1}$ are comparable, by using (1.1), we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}, a) \\ &= d(T x_{n(k)-1}, T x_{m(k)-1}, a) \\ &\leq \frac{1}{2} [d(x_{n(k)-1}, T x_{m(k)-1}, a) + d(x_{m(k)-1}, T x_{n(k)-1}, a)] \\ &\quad - \psi(d(x_{n(k)-1}, T x_{m(k)-1}, a), d(x_{m(k)-1}, T x_{n(k)-1}, a)) \\ &= \frac{1}{2} [d(x_{n(k)-1}, x_{m(k)}, a) + d(x_{m(k)-1}, x_{n(k)}, a)] \end{aligned}$$

$$- \psi(d(x_{n(k)-1}, xm(k), a), d(x_{m(k)-1}, xn(k), a)) \dots\dots\dots (1.20)$$

Taking the limit as $k \rightarrow \infty$ in (1.20) and using (1.16), (1.19) and the continuity of ψ , we have

$$\varepsilon \leq \frac{1}{2} (\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon) = \varepsilon - \psi(\varepsilon, \varepsilon) \leq \varepsilon.$$

It proves that $\psi(\varepsilon, \varepsilon) = 0$, that is, $\varepsilon = 0$. It is a contradiction. And this proves that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. It follows from the continuity of T that

$$z = \lim_{n \rightarrow \infty} xn+1 = \lim_{n \rightarrow \infty} Txn = Tz. \text{ Then } z \text{ is a fixed point of } T.$$

The next result is another one for the existence of the fixed point of a weak C -contraction on a 2-metric space.

Theorem 1.3 Let (X, \leq, d) be a complete, partially ordered 2-metric space and $T : X \rightarrow X$ be a weak C -contraction such that:

1. T is non-decreasing.
1. If $\{x_n\}$ is non-decreasing such that $\lim_{n \rightarrow \infty} xn = x$, then $xn \leq x$ for all $n \in \mathbb{N}$.
3. There exists $x_0 \in X$ with $x_0 \leq Tx_0$.

Then T has a fixed point.

Proof As in the proof of Theorem 1.2, we have a Cauchy sequence $\{x_n\}$ with $\lim_{n \rightarrow \infty} x_n = z$ in X . We only have to prove that $Tz = z$. Since $\{x_n\}$ is non-decreasing and $\lim_{n \rightarrow \infty} x_n = z$, we have $x_n \leq z$ for all $n \in \mathbb{N}$. It follows from (1.1) that

$$\begin{aligned} d(xn+1, Tz, a) &= d(Txn, Tz, a) \\ &\leq \frac{1}{2} [d(xn, Tz, a) + d(z, Txn, a) - \psi(d(xn, Tz, a), d(z, Txn, a))] \\ &= \frac{1}{2} [d(xn, Tz, a) + d(z, xn+1, a) - \psi(d(xn, Tz, a), d(z, xn+1, a))] \dots\dots\dots (1.21) \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in (1.21), we have

$$\begin{aligned} d(z, Tz, a) &\leq \frac{1}{2} [d(z, Tz, a) + d(z, z, a) - \psi(d(z, Tz, a), d(z, z, a))] \\ &\leq \frac{1}{2} d(z, Tz, a) - \psi(d(z, Tz, a), 0) \\ &\leq \frac{1}{2} d(z, Tz, a). \end{aligned}$$

It implies that $d(z, Tz, a) = 0$ for all $a \in X$, that is, $Tz = z$.

So, we proved a sufficient condition for the uniqueness of the fixed point in Theorem 1.2 and Theorem 1.3.

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